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Gauge forms on $SU(2)$-bundles

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Abstract

The structure of differential forms on the bundle of connections $p : C(P) \to M$ of a principal $SU(2)$-bundle $\pi : P \to M$ which are invariant under the natural representation of the gauge algebra of P on connections is determined. The invariance under the Lie algebra of all infinitesimal automorphisms of P is also analyzed. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The main goal of this paper is to determine the structure of the algebra of gauge invariant differential forms on the bundle of connections $p : C(P) \to M$ of a principal $SU(2)$-bundle $\pi : P \to M$. It is proved that this algebra is generated over $p^*\Omega^\bullet(M)$ by a closed 4-form η_4 globally defined on $C(P)$. The cohomology class of η_4 in $H^4(C(P); \mathbb{R}) \cong H^4(M; \mathbb{R})$ is also proved to be $-4\pi^2$ times the Chern class $c_2(P)$ of the given bundle, but we should mention that η_4 provides more information than the Chern class. For example, if $\dim M \leq 3$, then $c_2(P) = 0$ (in fact, P is trivial in this case) but the form η_4 does not vanish on $C(P)$. If P is trivializable, $C(P)$ can be identified to the $\mathfrak{su}(2)$-valued covectors on M by using a trivialization $P \cong M \times SU(2)$; i.e., $C(P) \cong T^*M \otimes \mathfrak{su}(2)$. The bundle $T^*M \otimes \mathfrak{su}(2)$

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is endowed with a generalized Liouville form ω_M with values in $\mathfrak{su}(2)$. The relevant fact is that the determinant (taken in the Lie algebra $\mathfrak{su}(2)$) of the 2-form $d\omega_M + \omega_M \wedge \omega_M$ does not depend on the particular trivialization chosen. In this way we obtain a differential form η_4 defined on $C(P)$ for an arbitrary $SU(2)$-bundle P, not necessarily trivial. In [9,10] the bundle of connections of a $U(1)$-bundle has been endowed with a symplectic structure which coincides with that of T^*M in the trivial case and it is proved that the algebra of gauge invariant forms is generated by the corresponding symplectic form. From this point of view, the results below can be considered as an extension from the group S^1 to S^3.

The gauge algebra of P is defined to be the Lie algebra $\mathrm{gau}P$ of $SU(2)$-invariant π-vertical vector fields of P. More generally, we think of the Lie algebra $\mathrm{aut}P$ of all $SU(2)$-invariant vector fields of P as being the "infinitesimal automorphisms" of P. Hence, as the automorphisms of P acts on connections, we obtain a natural Lie algebra representation from $\mathrm{aut}P$ into the vector fields of $C(P)$, which we denote by $X \mapsto X_C$. Then, a differential form Ω_r on $C(P)$ is said to be $\mathrm{aut}P$-invariant (resp. gauge invariant) if $L_{X_C}\Omega_r = 0$, for every $X \in \mathrm{aut}P$ (resp. for every $X \in \mathrm{gau}P$). In order to state the basic results the technique is first to solve the problem on J^1P and then to go down onto the bundle of connections by using the natural identification $(J^1P)/G \cong C(P)$. Moreover, gauge invariance on J^1P is of interest by itself as the differential forms invariant under the representation of $\mathrm{gau}P$ into J^1P are shown to be generated by the standard contact forms and their exterior differentials.

The present work was initially originated from the geometric version of Utiyama's theorem ([3–5,7]) which classifies Lagrangian densities invariant under the gauge algebra representation. Due to the importance of this result in describing the geometry of gauge theories it seems reasonable to analyze it in full generality on a purely geometric setting.

2. Definitions and preliminaries

2.1. Automorphisms and the gauge group

An automorphism of a principal G-bundle $\pi : P \to M$ is an equivariant diffeomorphism $\Phi : P \to P$; i.e., Φ is a diffeomorphism such that $\Phi(u \cdot g) = \Phi(u) \cdot g, \forall u \in P, \forall g \in G$. The set of all automorphisms of P is a group under the composition of maps which will be denoted by $\mathrm{Aut}P$. An automorphism $\Phi \in \mathrm{Aut}P$ induces a unique diffeomorphism on the base manifold $\phi : M \to M$, such that $\pi \circ \Phi = \phi \circ \pi$. If ϕ is the identity map, then Φ is said to be a gauge transformation or even a bundle automorphism (cf. [4, 3.2.1; 8, III.35; 14, I.B]). The set of all gauge transformations is a subgroup $\mathrm{Gau}P \subset \mathrm{Aut}P$, which is called the gauge group of the given bundle. In the case of the trivial bundle $\mathrm{pr}_1 : M \times G \to M$, it is easily checked that every automorphism Φ can be written as

$$\Phi(x, g) = (\phi(x), \psi(x) \cdot g), \quad x \in M, \ g \in G, \tag{2.1}$$

where $\phi : M \to M$ is a diffeomorphism and $\psi : M \to G$ is a differentiable map. In particular, we have $\mathrm{Gau}(M \times G) \simeq C^\infty(M, G)$. Note however that this identification depends on the specific trivialization chosen.

2.2. G-invariant vector fields

A vector field $X \in \mathfrak{X}(P)$ is said to be G-invariant if $R_g \cdot X = X$, $\forall g \in G$, where R_g stands for the right translation by g. If Φ_t is the flow of a vector field $X \in \mathfrak{X}(P)$, then X is G-invariant if and only if $\Phi_t \in \mathrm{Aut}\,P$, $\forall t \in \mathbb{R}$. Because of this we think of G-invariant vector fields as being the 'Lie algebra' of the automorphism group $\mathrm{Aut}\,P$ and hence we denote the Lie subalgebra of G-invariant vector fields on P by $\mathrm{aut}\,P \subset \mathfrak{X}(P)$. Each G-invariant vector field on P is π-projectable. Similarly, a π-vertical vector field $X \in \mathfrak{X}(P)$ is G-invariant if and only if $\Phi_t \in \mathrm{Gau}\,P$, $\forall t \in \mathbb{R}$. Accordingly, we denote by $\mathrm{gau}\,P \subset \mathrm{aut}\,P$ the ideal of all π-vertical G-invariant vector fields on P, which will be called the gauge algebra of P.

Moreover, the group G acts on $T(P)$ by setting $X \cdot g = (R_g)_*(X), \forall X \in T(P), \forall g \in G$. The quotient $T(P)/G$ exists as a differentiable manifold and it is endowed with a vector bundle structure over M (see [1]), whose global sections can be naturally identified to $\mathrm{aut}\,P$; i.e., $\mathrm{aut}\,P \simeq \Gamma(M, T(P)/G)$. The gauge algebra of P can be identified to the adjoint bundle; i.e., the bundle associated to P by the adjoint representation of G on its Lie algebra $\mathfrak{g}$, denoted by $\pi_{\mathfrak{g}} : \mathrm{ad}\,P \to M$ (cf. [8, III.35; 13, I. Proposition 5.4]); that is, $\mathrm{ad}\,P = (P \times \mathfrak{g})/G$, where the action of G on $P \times \mathfrak{g}$ is given by

$$(u, A) \cdot g = (u \cdot g, \mathrm{Ad}_{g^{-1}}(A)), \quad \forall u \in P, \quad \forall A \in \mathfrak{g}, \quad \forall g \in G.$$

Hence, $\mathrm{gau}\,P \simeq \Gamma(M, \mathrm{ad}P)$. Given a pair $(u, A) \in (P \times \mathfrak{g})$ we shall denote its G-orbit in $\mathrm{ad}\,P$ by $((u, A))$. We also remark that the fibres $(\mathrm{ad}\,P)_x$ are endowed with a Lie algebra structure uniquely determined by the condition

$$[((u, A)), ((u, B))] = ((u, [A, B])), \quad \forall u \in \pi^{-1}(x), \forall A, B \in \mathfrak{g}, \tag{2.2}$$

where $[,]$ stands for the bracket in $\mathfrak{g}$, but this is no longer true for the fibres of $T(P)/G$. We obtain an exact sequence of vector bundles over M (the so-called Atiyah sequence, [1, Theorem 1]),

$$0 \to \mathrm{ad}\,P \to T(P)/G \xrightarrow{\pi_*} TM \to 0. \tag{2.3}$$

2.3. The bundle of connections

Let Γ be a connection on a principal G-bundle $\pi : P \to M$ and let $X^* \in \mathfrak{X}(P)$ be the horizontal lift (with respect to Γ) of a vector field $X \in \mathfrak{X}(M)$ (cf. [13, Chapter II, Section 1]). As is well-known (cf. [13, II.Proposition 1.2]) the horizontal lift X^* is a G-invariant vector field on P projecting onto X. Hence we have a splitting of (2.3),

$$\sigma_\Gamma : TM \to T(P)/G, \quad \sigma_\Gamma(X) = X^*. \tag{2.4}$$

Conversely, any splitting $\sigma : TM \to T(P)/G$ of the Atiyah sequence (i.e., σ is a vector bundle homomorphism such that $\pi_* \circ \sigma = 1_{TM}$) is induced from a unique connection on P; in other words, there is a natural one-to-one correspondence between connections on P and splittings of the Atiyah sequence. Accordingly, we define the bundle of connections $p : C(P) \to M$ as the sub-bundle of $\mathrm{Hom}(TM, T(P)/G)$ determined by all $\mathbb{R}$-linear mappings $\lambda : T_xM \to (T(P)/G)_x$ such that $\pi_* \circ \lambda = 1_{T_xM}$ (e.g., see [5, Definition 4.5; 7; 10]). Connections on P can thus be identified to the global sections of $p : \mathcal{C}(P) \to M$. We denote by

$$\sigma_\Gamma : M \to \mathcal{C}(P) \tag{2.5}$$

the section of the bundle of connections tautologically induced by a connection Γ. An element $\lambda : T_xM \to (T(P)/G)_x$ of the bundle $\mathcal{C}(P)$ over a point $x \in M$ is nothing but a 'connection at a point x'; i.e., λ induces a complementary subspace H_u of the vertical subspace $V_u(P) \subset T_u(P)$ for every $u \in \pi^{-1}(x)$. If we add a linear mapping $h : T_xM \to (\mathrm{ad}P)_x$ to λ we obtain another element $\lambda' = h + \lambda \in \mathcal{C}(P)$, as $h \in \ker \pi_*$. In this way we can say that $\mathcal{C}(P)$ is an affine bundle modelled over the vector bundle $\mathrm{Hom}(TM, \mathrm{ad}P) \simeq T^*M \otimes \mathrm{ad}P$.

2.4. *SU*(2) *notations*

Throughout this paper we consider the standard basis of the Lie algebra $\mathfrak{su}(2)$ normalized by the factor $1/2$ (e.g., see [2, II.1, p.19; 15, 10.8-(10.94)]); i.e.,

$$B_1 = \frac{1}{2}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \quad B_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B_3 = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{2.6}$$

with $\mathrm{i} = \sqrt{-1}$. Remark that $2\mathrm{i}B_a$, $1 \le a \le 3$, are the Pauli matrices. From the formula (2.6) we obtain

$$[B_1, B_2] = B_3, \quad [B_2, B_3] = B_1, \quad [B_3, B_1] = B_2. \tag{2.7}$$

We identify $SU(2)$ to the 3-sphere $S^3 \subset \mathbb{C}^2$, as follows. Let $(y^0 + \mathrm{i}y^1, y^2 + \mathrm{i}y^3)$ be the standard coordinates in $\mathbb{C}^2$. Then, a matrix $g \in SU(2)$ can be uniquely written as

$$g = \begin{pmatrix} y^0(g) + \mathrm{i}y^1(g) & y^2(g) + \mathrm{i}y^3(g) \\ -y^2(g) + \mathrm{i}y^3(g) & y^0(g) - \mathrm{i}y^1(g) \end{pmatrix} \tag{2.8}$$

with

$$y^0(g)^2 + y^1(g)^2 + y^2(g)^2 + y^3(g)^2 = 1. \tag{2.9}$$

2.5. *Coordinates on* $\mathcal{C}(P)$

Let $\pi : P \to M$ be a principal $SU(2)$-bundle and let $(U; x^1, \dots, x^n)$ be a coordinate open domain in M such that P is trivial over U. For every $B \in \mathfrak{su}(2)$ we can thus define a one-parameter group of gauge transformations over U by setting $\varphi_t^B(x, g) = (x, \exp(tB) \cdot g)$, $x \in U$. Let us denote by $\tilde{B}$ the corresponding infinitesimal generator. Then $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{B}_3$ are a basis of sections for $\mathrm{ad}\pi^{-1}(U)$. As σ_Γ is a section of π_* in (2.3) there exist unique functions $A_j^a(\Gamma) \in C^\infty(U)$ such that

$$\sigma_\Gamma\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial x^j} - A_j^a(\Gamma)\tilde{B}_a, \quad 1 \le j \le n. \tag{2.10}$$

The functions $(x^j; A^a_j)$, $1 \le j \le n$, $1 \le a \le 3$, induce a coordinate system on $p^{-1}(U) = \mathcal{C}(\pi^{-1}U)$. Note that $\dim \mathcal{C}(P) = 4n$, with $n = \dim M$. Let A be the $\mathfrak{su}(2)$-valued 1-form on $p^{-1}(U)$ given by

$$A = A^a \otimes B_a = \frac{1}{2}\begin{pmatrix} \mathrm{i}A^1 & A^2 + \mathrm{i}A^3 \\ -A^2 + \mathrm{i}A^3 & -\mathrm{i}A^1 \end{pmatrix}, \tag{2.11}$$

where $A^a = A^a_j \, \mathrm{d}x^j$, $1 \le a \le 3$. Then, for every connection Γ on P the following local expression of the connection form holds true [15, 7.10, formulas (7.93), (7.96) and (7.101)]:

$$\omega_\Gamma = g^{-1}\,\mathrm{d}g + g^{-1}\cdot A(\Gamma)\cdot g, \tag{2.12}$$

where $A(\Gamma)$ stands for $\sigma^*_\Gamma(A)$.

2.6. *The fundamental representation*

Each $\Phi \in \mathrm{Aut}P$ acts on the connections of P as follows: given Γ, $\Gamma' = \Phi(\Gamma)$ is the connection corresponding to the connection form $\omega_{\Gamma'} = (\Phi^{-1})^*\omega_\Gamma$ (cf. [13, II.Proposition 6.2 (b)]). If $\Psi \in \mathrm{Aut}P$ is another automorphism, then $(\Psi \circ \Phi)(\Gamma) = \Psi(\Phi(\Gamma))$. For each $\Phi \in \mathrm{Aut}P$ there exists a unique diffeomorphism

$$\Phi_\mathcal{C} : \mathcal{C}(P) \to \mathcal{C}(P) \tag{2.13}$$

such that $p \circ \Phi_\mathcal{C} = \phi \circ p$, where $\phi : M \to M$ is the diffeomorphism induced from Φ, and $\Phi_\mathcal{C} \circ \sigma_\Gamma = \sigma_{\Phi(\Gamma)}$, for every connection Γ on P. In this way we obtain a group homomorphism $\mathrm{Aut}P \to \mathrm{Diff}\mathcal{C}(P)$. If Φ_t is the flow of a G-invariant vector field $X \in \mathrm{aut}P$, then $(\Phi_t)_\mathcal{C}$ is a one-parameter group in $\mathcal{C}(P)$ and the corresponding infinitesimal generator will be denoted by $X_\mathcal{C}$. In this way we obtain a Lie algebra representation

$$\mathrm{aut}P \to \mathfrak{X}(\mathcal{C}(P)), \quad X \mapsto X_\mathcal{C}, \tag{2.14}$$

which will be called the fundamental representation of infinitesimal automorphisms of P on the bundle of connections. Notice that X and $X_\mathcal{C}$ both are projectable onto the same vector field of M. By using a coordinate domain $(U; x^1, \ldots, x^n)$ in M and the basis $\tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ of $\mathrm{ad}\pi^{-1}(U)$ introduced in Section 2.5, is immediate that each $X \in \mathrm{aut}\pi^{-1}(U)$ can be written as

$$X = f_j \frac{\partial}{\partial x^j} + g^a \tilde{B}_a, \quad f_j, g^a \in C^\infty(U), \tag{2.15}$$

and as a simple computation shows, we have

$$X_\mathcal{C} = f_j \frac{\partial}{\partial x^j} - \underset{123}{\mathfrak{S}}\left(\frac{\partial g^1}{\partial x^j} + \frac{\partial f_i}{\partial x^j}A^1_i + g^3 A^2_j - g^2 A^3_j\right)\frac{\partial}{\partial A^1_j}, \tag{2.16}$$

where $\mathfrak{S}$ stands for the cyclic sum. In particular, if X is an infinitesimal gauge transformation, then $f_j = 0$ and we obtain

$$X_\mathcal{C} = -\underset{123}{\mathfrak{S}}\left(\frac{\partial g^1}{\partial x^j} + g^3 A^2_j - g^2 A^3_j\right)\frac{\partial}{\partial A^1_j}. \tag{2.17}$$

A differential form ω_r on $\mathcal{C}(P)$ of arbitrary degree $0 \le r \le 4n = \dim \mathcal{C}(P)$ is said to be gau$P$-invariant (resp. aut$P$-invariant) if for every $X \in \operatorname{gau} P$ (resp. for every $X \in \operatorname{aut} P$) we have $L_{X_C}\omega_r = 0$. Usually, gau$P$-invariant differential forms are called gauge invariant forms. We denote by $\mathcal{I}_{\operatorname{gau} P}$ (resp. by $\mathcal{I}_{\operatorname{aut} P}$) the set of gau$P$-invariant differential forms (resp. aut$P$-invariant differential forms). Notice that $\mathcal{I}_{\operatorname{gau} P}$ is a $\mathbb{Z}$-graded algebra over $\Omega^\bullet(M)$ and $\mathcal{I}_{\operatorname{aut} P} \subset \mathcal{I}_{\operatorname{gau} P}$ is a subalgebra.

3. Statement of the main results

First, let us consider the trivial bundle $\mathrm{pr}_1 : M \times SU(2) \to M$. We can identify its bundle of connections with $\mathfrak{su}(2)$-valued covectors, i.e., $\mathcal{C}(M \times SU(2)) \simeq T^*M \otimes \mathfrak{su}(2)$, by means of the one-to-one correspondence $\Gamma \leftrightarrow \omega_\Gamma \leftrightarrow A(\Gamma)$ stated in the formula (2.12). Moreover, the bundle $T^*M \otimes \mathfrak{su}(2)$ is endowed with a canonical $\mathfrak{su}(2)$-valued 1-form ω_M which generalizes the Liouville form on the cotangent bundle, defined by $\omega_M(X) = w(p_*X)$, where X is a tangent vector at $w \in T^*M \otimes \mathfrak{su}(2)$. In terms of the coordinate system $(x^j; A_j^a)$, $1 \le j \le n$, $1 \le a \le 3$, on $p^{-1}(U)$ introduced in (2.10) it is obvious that the local expression of ω_M is

$$\omega_M = A_j^a \, \mathrm{d}x^j \otimes B_a. \tag{3.1}$$

Note that ω_M is p-horizontal and that for every connection Γ we (tautologically) have $\sigma_\Gamma^*\omega_M = A(\Gamma)$ (cf. formula (2.11)). Let us see how ω_M changes in making a gauge transformation Φ of the trivial bundle. If $\Phi(x, g) = (x, \psi(x) \cdot g)$, with $\psi : M \to SU(2)$ (cf. (2.1)), then from the formula (2.12) we obtain

$$\omega_{\Gamma'} = (\Phi^{-1})^*\omega_\Gamma = g^{-1}\,\mathrm{d}g + g^{-1} \cdot (\psi\, \mathrm{d}\psi^{-1} + \psi \cdot A(\Gamma) \cdot \psi^{-1}) \cdot g,$$
$$\Gamma' = \Phi_C(\Gamma).$$

Hence for every connection Γ, we have

$$\sigma_\Gamma^*(\Phi_C^*\omega_M) = \sigma_{\Gamma'}^*\omega_M = A(\Gamma') = \psi\, \mathrm{d}\psi^{-1} + \psi \cdot A(\Gamma) \cdot \psi^{-1}$$
$$= \sigma_\Gamma^*(\psi\, \mathrm{d}\psi^{-1} + \psi \cdot \omega_M \cdot \psi^{-1}).$$

As ω_M is horizontal we conclude that $\Phi_C^*\omega_M = \psi\, \mathrm{d}\psi^{-1} + \psi \cdot \omega_M \cdot \psi^{-1}$. Therefore,

$$\Phi_C^*(\mathrm{d}\omega_M + \omega_M \wedge \omega_M) = \psi \cdot (\mathrm{d}\omega_M + \omega_M \wedge \omega_M) \cdot \psi^{-1}. \tag{3.2}$$

Next, let us consider an arbitrary $SU(2)$-bundle $\pi : P \to M$ and let U be an open domain in M over which P is trivial. Let us choose a trivialization $\Psi : \pi^{-1}(U) \to U \times SU(2)$. We define a 4-form η_4^Ψ on $\mathcal{C}(\pi^{-1}(U))$ as follows:

$$\eta_4^\Psi = \Psi_C^*(\det(\mathrm{d}\omega_U + \omega_U \wedge \omega_U)) \quad \text{(cf. (2.13))},$$

where det : $\mathfrak{su}(2) \to \mathbb{R}$ is the determinant function on the Lie algebra. We prove that there exists a unique global 4-form η_4 on $\mathcal{C}(P)$ such that

$$(\eta_4)|_{p^{-1}U} = \Psi_C^*(\det(\mathrm{d}\omega_U + \omega_U \wedge \omega_U)). \tag{3.3}$$

To do this, it suffices to check that if $\Psi' : \pi^{-1}(U') \to U' \times SU(2)$ is another trivialization on an overlapping domain U' we have

$$(\eta_4^{\Psi})|_{p^{-1}(U\cap U')} = (\eta_4^{\Psi'})|_{p^{-1}(U\cap U')}.$$

In fact, $\Phi = \Psi' \circ \Psi^{-1} : U \cap U' \times SU(2) \to U \cap U' \times SU(2)$ is a gauge transformation and from formula (3.2) we obtain

$$\begin{aligned}
&(\eta_4^{\Psi'})|_{p^{-1}(U\cap U')} \\
&\quad = (\Phi_C \circ \Psi_C)^*(\det(\mathrm{d}\omega_U + \omega_U \wedge \omega_U)) \\
&\quad = \Psi_C^*(\Phi_C^*(\det(\mathrm{d}\omega_U + \omega_U \wedge \omega_U))) \\
&\quad = \Psi_C^*(\det(\Phi_C^*(\mathrm{d}\omega_U + \omega_U \wedge \omega_U))) \\
&\quad = \Psi_C^*(\det(\mathrm{d}\omega_U + \omega_U \wedge \omega_U)) \\
&\quad = (\eta_4^{\Psi})|_{p^{-1}(U\cap U')}.
\end{aligned}$$

By using formulas (2.11), (3.1) and (3.3) one obtains the local expression of η_4 on an induced coordinate system $(x^j; A_j^a)$ (cf. Section 2.5),

$$\eta_4 = \tfrac{1}{4} \underset{123}{\mathfrak{S}} (\mathrm{d}A_i^1 \wedge \mathrm{d}x^i \wedge \mathrm{d}A_j^1 \wedge \mathrm{d}x^j + 2A_j^2 A_k^3\, \mathrm{d}x^j \wedge \mathrm{d}x^k \wedge \mathrm{d}A_i^1 \wedge \mathrm{d}x^i). \tag{3.4}$$

Also note that, as the above argument proves, for every $\Phi \in \mathrm{Gau}\,P$ we have

$$\Phi_C^*(\eta_4) = \eta_4, \tag{3.5}$$

and therefore, $\eta_4 \in \mathcal{I}_{\mathrm{gau}\,P}$.

Theorem 3.1. *Let $\pi : P \to M$ be a principal $SU(2)$-bundle. The algebra of gauge invariant differential forms on the bundle of connections $p : \mathcal{C}(P) \to M$ is generated over the algebra of differential forms on M by the 4-form η_4; i.e.,*

$$\mathcal{I}_{\mathrm{gau}\,P}(\mathcal{C}(P)) = (p^*\Omega^\bullet(M))[\eta_4].$$

Theorem 3.2. *Let $\pi : P \to M$ be a principal $SU(2)$-bundle. Assume M is connected. The algebra of* aut$P$*-invariant differential forms on the bundle of connections $p : \mathcal{C}(P) \to M$ is generated over $\mathbb{R}$ by the 4-form η_4; i.e.,*

$$\mathcal{I}_{\mathrm{aut}\,P}(\mathcal{C}(P)) = \mathbb{R}[\eta_4].$$

As $p : \mathcal{C}(P) \to M$ is an affine bundle, the projection p induces an isomorphism in the cohomology algebra, $p^* : H^\bullet(M; \mathbb{R}) \xrightarrow{\sim} H^\bullet(\mathcal{C}(P); \mathbb{R})$. Hence given a cohomology class

$[\omega_i] \in H^i(\mathcal{C}(P); \mathbb{R})$, there exists a unique cohomology class $[w_i] \in H^i(M; \mathbb{R})$ such that $p^*([w_i]) = [\omega_i]$. Then, we have

Theorem 3.3. *Let $\pi : P \to M$ be an arbitrary principal $SU(2)$-bundle. The cohomology class of η_4 in $H^4(\mathcal{C}(P); \mathbb{R})$ coincides with $-4\pi^2$ times $p^*(c_2(P))$, where $c_2(P)$ stands for the second Chern class of P.*

Remark 3.1. *If* $\dim M \leq 3$, *then every principal $SU(2)$-bundle $\pi : P \to M$ is trivial, as $SU(2) \simeq S^3$ is $(m-1)$-connected for each $m \leq$* $\dim M$ *and hence P admits a global section (cf.* [11, *Chapter* 2, *Theorem* 7.1]) *but the form η_4 does not vanish although its pull-back along every connection does vanish.*

4. Gauge invariance in J^1P

4.1. The identification $(J^1P)/G \simeq \mathcal{C}(P)$

Let $\pi : P \to M$ be an arbitrary principal G-bundle and let $\pi_1 : J^1P \to M$ be the 1-jet bundle of local sections of π. The group G acts (on the right) on J^1P by $j_x^1 s \cdot g = j_x^1(R_g \circ s)$, where s is a section of π defined on a neighbourhood of $x \in M$, $g \in G$ and R_g stands for the right translation. The quotient $(J^1P)/G$ exists as a fibred differentiable manifold over M and can be identified to the bundle of connections (see [6]). This fact is sometimes used to define $\mathcal{C}(P)$; e.g., see [3,12] for this approach. Let us briefly describe this identification. Let $q : J^1P \to \mathcal{C}(P)$ be the mapping defined as follows. Each local section s defines a retract $\Gamma_{s(x)} : T_{s(x)}P \to V_{s(x)}P = \ker(\pi_*)_{s(x)}$ of the inclusion $V_{s(x)}P \subset T_{s(x)}P$ by setting $\Gamma_{s(x)}(X) = X - s_*\pi_*(X)$. For every $u \in \pi^{-1}(x)$ there exists a unique $g \in G$ such that $u = s(x) \cdot g$ and we define $\Gamma_u : T_uP \to V_uP$ as $\Gamma_u = (R_g)_* \circ \Gamma_{s(x)} \circ (R_{g^{-1}})_*$. In this way, we obtain a 'connection Γ at x'; that is, an element of $\mathcal{C}(P)$ which only depends on $j_x^1 s$. Hence we set $q(j_x^1 s) = \Gamma$. It is not difficult to prove that q is a surjective submersion whose fibres are the orbits of G. Accordingly, $(J^1P)/G$ can be identified to $\mathcal{C}(P)$.

4.2. Infinitesimal contact transformations

Let X be a π-projectable vector field on P, let X' be its projection onto M and let Φ_t, ϕ_t be the flows of X, X', respectively. A flow can be defined on J^1P by the formula

$$\Phi_t^{(1)}(j_x^1 s) = j_{\phi_t(x)}^1(\Phi_t \circ s \circ \phi_{-t}).$$

If X is π-vertical (i.e., $X' = 0$ or even $\phi_t = \mathrm{id}_M$) then $\Phi_t^{(1)} = J^1(\Phi_t)$. We denote by $X^{(1)}$ the infinitesimal generator of the flow $\Phi_t^{(1)}$ which is called the infinitesimal contact transformation associated to X (or also the natural lift of X to the 1-jet bundle). We remark that the mapping $X \mapsto X^{(1)}$ is a Lie algebra monomorphism and that $X^{(1)}$ is π_{10}-projectable onto X, where $\pi_{10} : J^1P \to P$ stands for the canonical projection.

Proposition 4.1. *For every $\Phi \in \mathrm{Aut}P$ we have $q \circ \Phi^{(1)} = \Phi_C \circ q$ (cf. formula (2.13)). Accordingly, for every $X \in \mathrm{aut}P$ the vector field $X^{(1)}$ is q-projectable and its projection is X_C (cf. formula (2.14)).*

4.3. Contact forms on J^1P

Let $\pi : P \to M$ be a principal $SU(2)$-bundle. We define a $\mathfrak{su}(2)$-valued 1-form θ on J^1P as follows. For every $Y \in T_{j^1_x s}(J^1P)$ we have $q(j^1_x s)((\pi_{10})_* Y) \in V_{s(x)}P$. If $B^* \in \mathfrak{X}(P)$ is the fundamental vector field associated to $B \in \mathfrak{su}(2)$ (cf. [13, I.5]) we have an isomorphism $P \times \mathfrak{su}(2) \to VP$ given by $(u, B) \mapsto B^*_u$. Consequently, there exists a unique $B \in \mathfrak{su}(2)$ such that $q(j^1_x s)((\pi_{10})_* Y) = B^*_{s(x)}$. Then, we set $\theta(Y) = B$. Using the basis (2.6), we obtain $\theta = \theta^a \otimes B_a$, where θ^1, θ^2, θ^3 are ordinary global 1-forms on J^1P called the standard contact forms.

Proposition 4.2. *The valued θ form enjoys the following properties:*
(1) *For every $\Phi \in \mathrm{Gau}P$, we have $J^1(\Phi)^*\theta = \theta$.*
(2) *For every $B \in \mathfrak{su}(2)$, let $B^\bullet$ be the fundamental vector field associated to B under the action of $SU(2)$ on J^1P. Then, $L_{B^\bullet}\theta = [\theta, B]$.*

Proof. For every $Y \in T_{j^1_x s}(J^1P)$ we have $(J^1(\Phi)^*\theta)(Y) = \theta(J^1(\Phi)_* Y)$ and

$$(\pi_{10})_*(J^1(\Phi))_* Y = (\pi_{10} \circ J^1(\Phi))_* Y = (\Phi \circ \pi_{10})_* Y = \Phi_*((\pi_{10})_* Y).$$

Hence $\theta(J^1(\Phi)_* Y) = C$, where $C \in \mathfrak{g}$ is the element determined by

$$\Phi_*[(\pi_{10})_* Y - s_*\pi_*(\pi_{10})_* Y] = C^*_{\Phi(s(x))}.$$

Let $B \in \mathfrak{g}$ be the vector defined by $B^*_{s(x)} = (\pi_{10})_* Y - s_*\pi_*(\pi_{10})_* Y$. Hence $\theta(Y) = B$, and we have $C^*_{\Phi(s(x))} = \Phi_* B^*_{s(x)} = B^*_{\Phi(s(x))}$, thus proving (1). In order to state (2), let us first calculate $J^1(R_g)^*\theta$ for $g \in G$. We have

$$\begin{aligned}
&(\pi_{10})_* J^1(R_g)_* Y - (R_g \circ s)_*\pi_*(\pi_{10})_* J^1(R_g)_* Y \\
&\quad = (R_g)_*(\pi_{10})_* Y - (R_g \circ s)_*\pi_*(R_g)_*(\pi_{10})_* Y \\
&\quad = (R_g)_*[(\pi_{10})_* Y - s_*\pi_*(\pi_{10})_* Y] = (R_g)_* B^*_{s(x)} = (\mathrm{Ad}_{g^{-1}} B)^*.
\end{aligned}$$

Hence $(J^1(R_g)^*\theta)(Y) = (\mathrm{Ad}_{g^{-1}} \circ \theta)(Y)$. As the flow of $B^\bullet$ is $J^1(R_{\exp(tB)})$, we can conclude. □

4.4. Gauge forms on J^1P

A differential form ω_r on J^1P is said to be gauge invariant if $L_{X^{(1)}}\omega_r = 0$ for all $X \in \mathrm{gau}P$.

Remark 4.1. *From Proposition* 4.2 (1) *it follows that the standard contact forms are gauge invariant but, in fact, by using the formulas* (2.16), (4.2) *below and the standard formulas for jet prolongation it is not difficult to prove that the form* θ *is* aut$P$*-invariant indeed.*

Theorem 4.3. *The algebra of gauge invariant forms on* J^1P *is generated over* $\pi_1^*\Omega^\bullet(M)$ *by the forms* $(\theta^a, \mathrm{d}\theta^a)$, $1 \le a \le 3$.

Proof. For every open subset $U \subseteq M$ we set $\mathcal{A}'(U) = \pi_1^*\Omega^\bullet(U)[\theta^a, \mathrm{d}\theta^a]$. Let $\mathcal{A}(U)$ be the algebra of gauge invariant forms on $J^1(\pi^{-1}U)$. From Remark 4.1, we have $\mathcal{A}'(U) \subseteq \mathcal{A}(U)$. As $\mathcal{A}'$ and $\mathcal{A}$ are sheaves of algebras over M, it will suffice to prove that $\mathcal{A}'(U) = \mathcal{A}(U)$ for every small enough open subset. Hence we can assume that P is trivial, $P = M \times SU(2)$. In this case, we can identify J^1P to the submanifold of $J^1(M \times \mathbb{C}^2)$ given by the equations

$$\sum_{i=0}^{3}(y^i)^2 = 1, \quad \sum_{i=0}^{3} y^i y^i_j = 0 \quad (1 \le j \le n), \tag{4.1}$$

as follows taking derivatives in (2.9), where $(x^j, y^i; y^i_j)$, $0 \le i \le 3$, $1 \le j \le n$, is the coordinate system induced from (x^j, y^i) on $J^1(M \times \mathbb{C}^2)$; i.e., $y^i_j(j^1_x s) = (\partial(y^i \circ s)/\partial x^j)(x)$.

Let ω_r be a gauge invariant form on J^1P and let $s_0 : M \to P$ be the unit section: $s_0(x) = (x, 1)$, $\forall x \in M$. We claim that

$$(\omega_r)_{j^1_x s_0} \in \mathcal{A}'_{j^1_x s_0}, \forall x \in M \Longrightarrow (\omega_r)_{j^1_x s} \in \mathcal{A}'_{j^1_x s}, \quad \text{for every local section } s \text{ of } P.$$

In fact, let Φ be the gauge transformation given by $\Phi(x, g) = (x, \psi(x)g)$, where $s(x) = (x, \psi(x))$. We have $\Phi \circ s_0 = s$, and since $SU(2)$ is connected, there exists a one parameter group of gauge transformations Φ_t such that $\Phi_1 = \Phi$. Let X be the infinitesimal generator of Φ_t. As $L_{X^{(1)}}\omega_r = 0$, we have $J^1(\Phi_t)^*\omega_r = \omega_r$, $\forall t \in \mathbb{R}$; in particular for $t = 1$. Then,

$$(\omega_r)_{j^1_x s} = J^1(\Phi^{-1})^*((\omega_r)_{j^1_x s_0}),$$

and it suffices to take into account that

$$J^1(\Phi^{-1})^*\mathcal{A}'_{j^1_x s_0} = \mathcal{A}'_{j^1_x s}.$$

Accordingly, we only need to prove that every gauge invariant form ω_r belongs to $\mathcal{A}'$ along the section $j^1 s_0$. Moreover, as a simple computation shows, the standard contact forms have the following local expression

$$\begin{pmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{pmatrix} = 2 \begin{pmatrix} -y^1 & y^0 & y^3 & -y^2 \\ -y^2 & -y^3 & y^0 & y^1 \\ -y^3 & y^2 & -y^1 & y^0 \end{pmatrix} \begin{pmatrix} \mathrm{d}y^0 - y^0_j\,\mathrm{d}x^j \\ \mathrm{d}y^1 - y^1_j\,\mathrm{d}x^j \\ \mathrm{d}y^2 - y^2_j\,\mathrm{d}x^j \\ \mathrm{d}y^3 - y^3_j\,\mathrm{d}x^j \end{pmatrix}. \tag{4.2}$$

Set

$$\zeta^1 = \mathrm{d}\theta^1 + \theta^2 \wedge \theta^3, \quad \zeta^2 = \mathrm{d}\theta^2 + \theta^3 \wedge \theta^1, \quad \zeta^3 = \mathrm{d}\theta^3 + \theta^1 \wedge \theta^2. \tag{4.3}$$

Evaluating on the unit section we obtain

$$(\theta^a)_{j^1_x s_0} = 2(\mathrm{d}y^a)_{j^1_x s_0}; \quad (\zeta^a)_{j^1_x s_0} = 2(\mathrm{d}x^j \wedge \mathrm{d}y^a_j)_{j^1_x s_0}, \quad 1 \le a \le 3. \tag{4.4}$$

As $(\theta^a, \mathrm{d}\theta^a)$ and (θ^a, ζ^a) span the same algebra, it will suffice to prove that $(\omega_r)_{j^1_x s_0}$ can be written as a polynomial in (θ^a, ζ^a) with coefficients in $\Omega^\bullet(M)$.

As $y^0(j^1_x s_0) = 1, y^i(j^1_x s_0) = 0, 1 \le i \le 3, x \in M$, the functions $(x^j, y^i; y^i_j)$, $1 \le j \le n$, $1 \le i \le 3$, constitute a coordinate system on a neighbourhood N of the submanifold $\{j^1_x s_0 \mid x \in M\} \subset J^1(M \times SU(2)) \subset J^1(M \times \mathbb{C}^2)$. Hence any differential r-form defined on N can be uniquely written as

$$\omega_r = \sum_{|H|+|I|+|J|+|K|+|L|=r} f_{HIJKL}\, \mathrm{d}x^H \wedge (\mathrm{d}y^1)^{i_1} \wedge (\mathrm{d}y^2)^{i_2} \wedge (\mathrm{d}y^3)^{i_3} \wedge (\mathrm{d}y^1_1)^{j_1}$$
$$\wedge \cdots \wedge (\mathrm{d}y^1_n)^{j_n} \wedge (\mathrm{d}y^2_1)^{k_1} \wedge \cdots \wedge (\mathrm{d}y^2_n)^{k_n} \wedge (\mathrm{d}y^3_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}y^3_n)^{l_n},$$

where $f_{HIJKL} \in C^\infty(N)$, $\mathrm{d}x^H = (\mathrm{d}x^1)^{h_1} \wedge \cdots \wedge (\mathrm{d}x^n)^{h_n}$, $H = (h_1, \ldots, h_n)$, $I = (i_1, i_2, i_3)$, $J = (j_1, \ldots, j_n)$, $K = (k_1, \ldots, k_n)$, $L = (l_1, \ldots, l_n)$, with $|H| = h_1 + \cdots + h_n$, $|I| = i_1 + i_2 + i_3$, etc., and all indices $h_\alpha, i_1, i_2, i_3, j_\alpha, k_\alpha, l_\alpha$, $1 \le \alpha \le n$, belong to $\{0, 1\}$; *i.e.*, they are Boolean indices. Let us fix a point $x_0 \in M$. Set $x^j(x_0) = x^j_0$, $1 \le j \le n$. Let us consider the vector field X in (2.15) given by: $f_j = 0$, $1 \le j \le n$, $g^1 = (x^\alpha - x^\alpha_0)^2$, α being a fixed index, and $g^2 = g^3 = 0$. The infinitesimal contact transformation (cf. Section 4.2) associated to X in $J^1(M \times \mathbb{C}^2)$ is given by

$$X^{(1)} = X + \frac{1}{2}\left(\left(-y^1\frac{\partial g^1}{\partial x^j} - y^1_j g^1\right)\frac{\partial}{\partial y^0_j} + \left(y^0\frac{\partial g^1}{\partial x^j} + y^0_j g^1\right)\frac{\partial}{\partial y^1_j}\right.$$
$$\left. + \left(-y^3\frac{\partial g^1}{\partial x^j} - y^3_j g^1\right)\frac{\partial}{\partial y^2_j} + \left(y^2\frac{\partial g^1}{\partial x^j} + y^2_j g^1\right)\frac{\partial}{\partial y^3_j}\right).$$

Note that $X^{(1)}$ is tangent to $J^1(M \times SU(2))$ and its restriction $\bar{X}^{(1)}$ to this submanifold is the infinitesimal contact transformation associated to X in $J^1(M \times SU(2))$. From the definition of X, for every $f \in C^\infty(N)$ we obtain $\bar{X}^{(1)} f(j^1_{x_0} s_0) = 0$. Furthermore, from Proposition 4.2 (1) and (4.2), we have

$$(L_{\bar{X}^{(1)}}\theta^a)_{j^1_{x_0} s_0} = 2(L_{\bar{X}^{(1)}}\,\mathrm{d}y^a)_{j^1_{x_0} s_0} = 0, \quad 1 \le a \le 3,$$

and from a simple computation,

$$(L_{\bar{X}^{(1)}}\,\mathrm{d}y^1_j)_{j^1_{x_0} s_0} = \delta^\alpha_j (\mathrm{d}x^\alpha)^1_{j_{x_0}} s_0; \quad (L_{\bar{X}^{(1)}}\,\mathrm{d}y^a_j)^1_{j_{x_0}} s_0 = 0, \quad a = 2, 3.$$

Hence, taking into account that ω_r is gauge invariant,

$$0=(L_{\bar{X}^{(1)}}\omega_r)_{j^1_{x_0}s_0} = \left(\sum_{j_\alpha=1} f_{HIJKL}\, \mathrm{d}x^H \wedge (\mathrm{d}y^1)^{i_1} \wedge (\mathrm{d}y^2)^{i_2} \wedge (\mathrm{d}y^3)^{i_3} \right.$$
$$\wedge (\mathrm{d}y^1_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}y^1_{\alpha-1})^{j_{\alpha-1}} \wedge \mathrm{d}x^\alpha \wedge (\mathrm{d}y^1_{\alpha+1})^{j_{\alpha+1}} \wedge \cdots \wedge (\mathrm{d}y^1_n)^{j_n}$$
$$\left. \wedge (\mathrm{d}y^2_1)^{k_1} \wedge \cdots \wedge (\mathrm{d}y^2_n)^{k_n} \wedge (\mathrm{d}y^3_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}y^3_n)^{l_n} \right)_{j^1_{x_0}s_0}.$$

Therefore, we obtain $(h_\alpha = 0, j_\alpha = 1) \Rightarrow f_{HIJKL}(j^1_{x_0}s_0) = 0$. As α and x_0 are arbitrary, we can conclude that $f_{HIJKL} \circ j^1 s_0 = 0$ whenever an index α exists such that $j_\alpha = 1$ and $h_\alpha = 0$. Hence, along the unit section, the form ω_r can be rewritten as

$$(\omega_r)_{j^1 s_0} = \sum_{h_a+j_a<2} \{F_{HIJKL}\, \mathrm{d}x^H \wedge (\mathrm{d}y^1)^{i_1} \wedge (\mathrm{d}y^2)^{i_2} \wedge (\mathrm{d}y^3)^{i_3}$$
$$\wedge (\mathrm{d}x^1 \wedge \mathrm{d}y^1_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}x^n \wedge \mathrm{d}y^1_n)^{j_n}$$
$$\wedge (\mathrm{d}y^2_1)^{k_1} \wedge \cdots \wedge (\mathrm{d}y^2_1)^{k_n} \wedge (\mathrm{d}y^3_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}y^3_n)^{l_n}\}_{j^1 s_0}.$$

Moreover, let us consider the vector field $X \in \mathrm{gau}P$ given as follows: $g^1 = 2(x^1 - x^1_0)(x^\alpha - x^\alpha_0)$, $2 \le \alpha \le n$, $g^2 = g^3 = 0$. Its infinitesimal contact transformation, restricted to N, verifies

$$\bar{X}^{(1)} f(j^1_{x_0}s_0) = 0,\ \forall f \in C^\infty(N); \quad (L_{\bar{X}^{(1)}}\mathrm{d}y^a)_{j^1_{x_0}s_0} = 0,\ 1 \le a \le 3;$$
$$(L_{\bar{X}^{(1)}}dy^1_j)_{j^1_{x_0}s_0} = \delta^\alpha_j\, \mathrm{d}x^1 + \delta^1_j\, \mathrm{d}x^\alpha; \quad (L_{\bar{X}^{(1)}}\mathrm{d}y^a_j)_{j^1_{x_0}s_0} = 0,\ 2 \le a \le 3.$$

Evaluating the Lie derivative of ω_r at $j^1_{x_0}s_0$, we obtain

$$0 = \sum_{h_a+j_a<2,\, j_1=1} \left\{ F_{HIJKL}\, \mathrm{d}x^H \wedge (\mathrm{d}sy^1)^{i_1} \wedge (\mathrm{d}y^2)^{i_2} \wedge (\mathrm{d}y^3)^{i_3} \right.$$
$$\wedge (\mathrm{d}x^1 \wedge \mathrm{d}x^\alpha) \wedge \cdots \wedge (\mathrm{d}x^n \wedge \mathrm{d}y^1_n)^{j_n}$$
$$\wedge (\mathrm{d}y^2_1)^{k_1} \wedge \cdots \wedge (\mathrm{d}y^2_1)^{k_n} \wedge (\mathrm{d}y^3_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}y^3_n)^{l_n}$$
$$+ \sum_{h_a+j_a<2,\, j_\alpha=1} F_{HIJKL}\, \mathrm{d}x^H \wedge (\mathrm{d}y^1)^{i_1} \wedge (\mathrm{d}y^2)^{i_2} \wedge (\mathrm{d}y^3)^{i_3}$$
$$\wedge (\mathrm{d}x^1 \wedge \mathrm{d}y^1_1)^{j_1} \wedge \cdots \wedge (\mathrm{d}x^{\alpha-1} \wedge \mathrm{d}y^1_{\alpha-1})^{j_{\alpha-1}} \wedge (\mathrm{d}x^\alpha \wedge \mathrm{d}x^1)$$
$$\wedge (\mathrm{d}x^{\alpha+1} \wedge \mathrm{d}y^1_{\alpha+1})^{j_{\alpha+1}} \wedge \cdots \wedge (\mathrm{d}x^n \wedge \mathrm{d}y^1_n)^{j_n}$$
$$\left. \wedge (\mathrm{d}y^2_1)^{k_1} \wedge \cdots \wedge (\mathrm{d}y^2_1)^{k_n} \wedge (\mathrm{d}y^3_1)^{l_1} \wedge \cdots \wedge (\mathrm{d}y^3_n)^{l_n} \right\}_{j^1_{x_0}s_0}.$$

The above equation implies $F_{HIJKL} = F_{HI\tilde{J}KL}$, whenever $j_1 = \tilde{j}_\alpha$, $j_\alpha = \tilde{j}_1$ and $j_s = \tilde{j}_s$, $s \ne 1, \alpha$. Moving the indices 1 and α, and the point x_0, we can ensure that if a term $\omega_{r-2} \wedge \mathrm{d}x^j \wedge \mathrm{d}y^1_j$ appears in the expression of ω_r, it comes from the bigger summand

$\omega_{r-2} \wedge (dx^1 \wedge dy_1^1 + \cdots + dx^n \wedge dy_n^1) = \omega_{r-2} \wedge \zeta_1$. Substituting the indices 2 and 3 succesively for the index 1 in the definition of $X \in \mathrm{gau}P$, we can similarly conclude that ω_r can be written as

$$(\omega_r)_{j^1_x s_0} = \left(\sum f_{HIJKL} dx^H \wedge (\theta^1)^{i_1} \wedge (\theta^2)^{i_2} \wedge (\theta^3)^{i_3} \wedge (\zeta^1)^j \wedge (\zeta^2)^k \wedge (\zeta^3)^l\right)_{j^1_x s_0},$$

thus finishing the proof. □

Corollary 4.4. *Assume M is connected. The algebra of* aut$P$*-invariant forms on J^1P is generated over $\mathbb{R}$ by the forms $(\theta^a, d\theta^a)$, $1 \le a \le 3$.*

Proof. This follows from Theorem 4.3 taking into account Remark 4.1. □

5. Proof of Theorem 3.1

We first remark that a differential form ω_r on $C(P)$ is gauge invariant if and only if $q^*\omega_r$ is gauge invariant on J^1P with respect to the action of gau$P$ in J^1P (see Proposition 4.1) and that the differential forms in $q^*\mathcal{I}_{\mathrm{gau}P}(C(P))$ can be identified to the differential forms Ω_r on J^1P which are gauge invariant and such that

$$\text{(i) } i_{B^\bullet}\Omega_r = 0; \quad \text{(ii) } L_{B^\bullet}\Omega_r = 0, \ \forall B \in \mathfrak{su}(2),$$

as conditions (i) and (ii) are equivalent to saying that Ω_r is q-projectable onto the bundle of connections.

Let Ω_r be a gauge invariant form on J^1P. According to Theorem 4.3, Ω_r can be written as

$$\Omega_r = \sum_{i,\alpha} \omega_{i,\alpha} \wedge (\theta^1)^{i_1} \wedge (\theta^2)^{i_2} \wedge (\theta^3)^{i_3} \wedge (\zeta^1)^{\alpha_1} \wedge (\zeta^2)^{\alpha_2} \wedge (\zeta^3)^{\alpha_3}, \tag{5.1}$$

where $i = (i_1, i_2, i_3) \in \{0,1\}^3$, $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, the forms θ^a, ζ^a are defined in Section 4.3 and in the formula (4.3) respectively, and $\omega_{i,\alpha}$ is a differential form on $p^*\Omega^\bullet(M)$ of degree $r - (i_1 + i_2 + i_3) - 2(\alpha_1 + \alpha_2 + \alpha_3)$. Note that ζ^a can be substituted for $d\theta^a$ as $(\theta^a, d\theta^a)$ and (θ^a, ζ^a) span the same algebra. By imposing the condition (i) above in (5.1) and taking into account that $\theta^a(B_b^\bullet) = \delta_b^a$ (or equivalenty, $\theta(B^\bullet) = B$), and $i_{B_b^\bullet}\zeta^a = 0$, as follows from the very definition of ζ^a and the formula in Proposition 4.2 (2), we have

$$\begin{aligned} 0 = i_{B_h^\bullet}\Omega_r = &\sum_{i_2,i_3,\alpha} \delta_1^h \omega_{1,i_2,i_3,\alpha} \wedge (\theta^2)^{i_2} \wedge (\theta^3)^{i_3} \wedge (\zeta^1)^{\alpha_1} \wedge (\zeta^2)^{\alpha_2} \wedge (\zeta^3)^{\alpha_3} \\ &- \sum_{i_1,i_3,\alpha} \delta_2^h \omega_{i_1,1,i_3,\alpha} \wedge (\theta^1)^{i_1} \wedge (\theta^3)^{i_3} \wedge (\zeta^1)^{\alpha_1} \wedge (\zeta^2)^{\alpha_2} \wedge (\zeta^3)^{\alpha_3} \\ &+ \sum_{i_1,i_2,\alpha} \delta_3^h \omega_{i_1,i_2,1,\alpha} \wedge (\theta^1)^{i_1} \wedge (\theta^2)^{i_2} \wedge (\zeta^1)^{\alpha_1} \wedge (\zeta^2)^{\alpha_2} \wedge (\zeta^3)^{\alpha_3}. \end{aligned}$$

Hence $\omega_{i,\alpha} = 0$ for every $i \neq (0,0,0)$, and consequently,

$$\Omega_r = \sum_{\alpha} \omega_\alpha \wedge (\zeta^1)^{\alpha_1} \wedge (\zeta^2)^{\alpha_2} \wedge (\zeta^3)^{\alpha_3}. \tag{5.2}$$

Let us now impose the condition (ii) above in (5.2). First, as a simple computation shows, we have

$$\begin{array}{lll} L_{B_1^\bullet}\zeta^1 = 0, & L_{B_1^\bullet}\zeta^2 = \zeta^3, & L_{B_1^\bullet}\zeta^3 = -\zeta^2, \\ L_{B_2^\bullet}\zeta^1 = -\zeta^3, & L_{B_2^\bullet}\zeta^2 = 0, & L_{B_2^\bullet}\zeta^3 = \zeta^1, \\ L_{B_3^\bullet}\zeta^1 = \zeta^2, & L_{B_3^\bullet}\zeta^2 = -\zeta^1, & L_{B_3^\bullet}\zeta^3 = 0. \end{array}$$

Hence,

$$\begin{aligned} 0 &= L_{B_1^\bullet}\Omega_r \\ &= \sum_{\alpha} \omega_\alpha \wedge (\zeta^1)^{\alpha_1} \wedge (\alpha_2(\zeta^2)^{\alpha_2-1} \wedge (\zeta^3)^{\alpha_3+1} - \alpha_3(\zeta^2)^{\alpha_2+1} \wedge (\zeta^3)^{\alpha_3-1}), \end{aligned} \tag{5.3}$$

and similarly for B_2 and B_3. Let us assume the following.

Lemma 5.1. *The algebra generated by $\zeta^1, \zeta^2, \zeta^3$ over $p^*\Omega^\bullet(M)$ is the quotient of the polynomial algebra $p^*\Omega^\bullet(M)[t_1, t_2, t_3]$ modulo the ideal generated by the elements of the form $\omega_r t_1^{m_1} t_2^{m_2} t_3^{m_3}$, $r + m_1 + m_2 + m_3 > n = \dim M$, $\omega_r \in p^*\Omega^r(M)$.*

Then, the proof of the theorem can be concluded as follows. The coefficient of the term $(\zeta^1)^{\sigma_1} \wedge (\zeta^2)^{\sigma_2} \wedge (\zeta^3)^{\sigma_3}$, $\deg \omega_\sigma + \sigma_1 + \sigma_2 + \sigma_3 \leq n$, in formula (5.3) is $(\sigma_2+1)\omega_{\sigma_1,\sigma_2+1,\sigma_3-1} - (\sigma_3+1)\omega_{\sigma_1,\sigma_2-1,\sigma_3+1}$, which must vanish by virtue of the lemma. Proceeding similarly with the other two cases, we obtain

$$(\sigma_2 + 1)\omega_{\sigma_1,\sigma_2+1,\sigma_3-1} = (\sigma_3 + 1)\omega_{\sigma_1,\sigma_2-1,\sigma_3+1}, \tag{5.4}$$

$$(\sigma_1 + 1)\omega_{\sigma_1+1,\sigma_2,\sigma_3-1} = (\sigma_3 + 1)\omega_{\sigma_1-1,\sigma_2,\sigma_3+1}, \tag{5.5}$$

$$(\sigma_1 + 1)\omega_{\sigma_1+1,\sigma_2-1,\sigma_3} = (\sigma_2 + 1)\omega_{\sigma_1-1,\sigma_2+1,\sigma_3}. \tag{5.6}$$

Letting $\sigma_1 = 0$ in (5.6), we deduce $\omega_{1,\sigma_2-1,\sigma_3} = 0$. By recurrence on σ_1 in (5.6), we conclude that $\omega_{\sigma_1,\sigma_2,\sigma_3} = 0$ if σ_1 is an odd integer. Using (5.4) and (5.5) in the same way, we have $\omega_{\alpha_1,\alpha_2,\alpha_3} = 0$ if any index α_i is odd. Set $\alpha_i = 2\beta_i$ and $\varphi_\beta = \varphi_{\beta_1,\beta_2,\beta_3} = \omega_{2\beta_1,2\beta_2,2\beta_3} = \omega_\alpha$. Then

$$\Omega_r = \sum_{\beta} \varphi_\beta \wedge ((\zeta^1)^2)^{\beta_1} \wedge ((\zeta^2)^2)^{\beta_2} \wedge ((\zeta^3)^2)^{\beta_3},$$

and formulas (5.4), (5.5) and (5.6) become

$$\beta_2\varphi_{\beta_1,\beta_2,\beta_3-1} = \beta_3\varphi_{\beta_1,\beta_2-1,\beta_3}, \tag{5.7}$$

$$\beta_3\varphi_{\beta_1-1,\beta_2,\beta_3} = \beta_1\varphi_{\beta_1,\beta_2,\beta_3-1}, \tag{5.8}$$

$$\beta_1\varphi_{\beta_1,\beta_2-1,\beta_3} = \beta_2\varphi_{\beta_1-1,\beta_2,\beta_3}. \tag{5.9}$$

By induction on β_1 and using the formula (5.9), it is easily checked that

$$\varphi_{\beta_1,\beta_2,0} = \frac{(\beta_1+\beta_2)!}{\beta_1!\beta_2!}\varphi_{\beta_1+\beta_2,0,0}, \quad \beta_1+\beta_2 \le n,$$

and then, by induction on β_3 and again using formulas (5.7), (5.8) and (5.9), we finally obtain

$$\varphi_{\beta_1,\beta_2,\beta_3} = \frac{(\beta_1+\beta_2+\beta_3)!}{\beta_1!\beta_2!\beta_3!}\varphi_{\beta_1+\beta_2+\beta_3,0,0}.$$

Accordingly, from Leibniz's formula we have

$$\begin{aligned}
\Omega_r &= \sum_{\alpha} \omega_\alpha \wedge (\zeta^1)^{\alpha_1} \wedge (\zeta^2)^{\alpha_2} \wedge (\zeta^3)^{\alpha_3} \\
&= \sum_{2(\beta_1+\beta_2+\beta_3)\le r} \varphi_\beta \wedge ((\zeta^1)^2)^{\beta_1} \wedge ((\zeta^2)^2)^{\beta_2} \wedge ((\zeta^3)^2)^{\beta_3} \\
&= \sum_{k=0}^{[r/2]} \sum_{\beta_1+\beta_2+\beta_3=k} \frac{k!}{\beta_1!\beta_2!\beta_3!}\varphi_{k,0,0} \wedge ((\zeta^1)^2)^{\beta_1} \wedge ((\zeta^2)^2)^{\beta_2} \wedge ((\zeta^3)^2)^{\beta_3} \\
&= \sum_{k=0}^{[r/2]} \varphi_{k,0,0} \wedge ((\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2)^k.
\end{aligned}$$

Hence we only need to prove the following identity:

$$(\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2 = 4(q^*\eta_4). \tag{5.10}$$

To do this, we first remark that the problem being local, we can assume the bundle is trivial $P = M \times SU(2)$. Moreover, as both sides of (5.10) are gauge invariant forms, behaving as in the begining of the proof of Theorem 4.3, it suffices to prove that the formula (5.10) holds true along the 1-jet of the unit section $s_0 = (1_M, 1)$. First, let us calculate the equations of the quotient map $q : J^1P \to C(P)$ in terms of the natural coordinate systems $(x^j, y^i; y^i_j)$, $0 \le i \le 3$, $1 \le j \le n$ (with the constrains (4.1)); $(x^j; A^a_j)$, $1 \le j \le n$, $1 \le a \le 3$, in J^1P, $C(P)$, respectively. Let $\Gamma = q \circ j^1 s$ be the connection attached to a local section s. By imposing that Γ vanishes on its own horizontal lift given by formula (2.10) we obtain the expression of $A^a_j(\Gamma)$ in terms of the jet coordinates; that is,

$$\begin{pmatrix} A^1_j \\ A^2_j \\ A^3_j \end{pmatrix} = -\frac{2}{y^0}\begin{pmatrix} (y^0)^2+(y^1)^2 & y^1y^2 - y^0y^3 & y^1y^3 - y^0y^2 \\ y^0y^3 + y^1y^2 & (y^0)^2+(y^2)^2 & y^2y^3 - y^0y^1 \\ y^1y^3 - y^0y^2 & y^0y^1 + y^2y^3 & (y^0)^2+(y^3)^2 \end{pmatrix}\begin{pmatrix} y^1_j \\ y^2_j \\ y^3_j \end{pmatrix}, \tag{5.11}$$

over the open subset $y^0 \ne 0$, which contains the graph $\{j^1_x s_0 | x \in M\}$ of the unit section.

Moreover, restricting $q^*\eta_4$ to $j^1 s_0$, from the formulas (3.4) and (5.11) we have

$$\begin{aligned}
(q^*\eta_4)_{j^1_x s_0} &= \tfrac{1}{4}(q^* \underset{123}{\mathfrak{S}} (\mathrm{d}A^1_i \wedge \mathrm{d}x^i \wedge \mathrm{d}A^1_j \wedge \mathrm{d}x^j))_{j^1_x s_0} \\
&= \tfrac{1}{4}(q^* \underset{123}{\mathfrak{S}} (\mathrm{d}A^1_j \wedge \mathrm{d}x^j)^2)_{j^1_x s_0}.
\end{aligned}$$

As $(\mathrm{d}A_j^a)_{j_x^1 s_0} = -2(\mathrm{d}y_j^a)_{j_x^1 s_0}$, $1 \le a \le 3$, $1 \le j \le n$, from the second formula in (4.4) we conclude.

Proof of Lemma 5.1. We first remark that ζ^a is gauge invariant as follows from its very definition in formula (4.3) and (1) in Proposition 4.2. We have a natural epimorphism of graded algebras,

$$E : p^*\Omega^\bullet(M)[t^1, t^2, t^3] \to p^*\Omega^\bullet(M)[\zeta^1, \zeta^2, \zeta^3],\ E(t^a) = \zeta^a, 1 \le a \le 3.$$

We claim that ker E is generated by the elements in the statement. First, we prove that $\omega_r t_1^{m_1} t_2^{m_2} t_3^{m_3} \in \ker E$ for $r + m_1 + m_2 + m_3 > n = \dim M$. To do this, behaving as in the proof of Theorem 4.3, we only need to prove that $\omega_r \wedge (\zeta^1)^{m_1} \wedge (\zeta^2)^{m_2} \wedge (\zeta^3)^{m_3}$ vanishes along the submanifold $\{j_x^1 s_0 \mid x \in M\} \subset J^1 P$ for $r + m_1 + m_2 + m_3 > n$, where s_0 is the unit section of the trivial bundle. This directly follows from the expression of ζ^a along $j^1 s_0$ in the formula (4.4).

Conversely, if

$$\sum_{r+2(m_1+m_2+m_3)=R} \omega_{r,m} t_1^{m_1} t_2^{m_2} t_3^{m_3}, \quad \omega_{r,m} \in p^*\Omega^r(M),\ r + m_1 + m_2 + m_3 \le n$$

lies in ker E, then we have

$$0 = \sum_{r+2(m_1+m_2+m_3)=R} (\omega_{r,m})_x \wedge ((\zeta^1)^{m_1} \wedge (\zeta^2)^{m_2} \wedge (\zeta^3)^{m_3})_{j_x^1 s_0}, \tag{5.12}$$

and again using the expression of ζ^a in the formula (4.4) we conlude that all $(\omega_{r,m})_x$ must vanish. In fact, if $(\omega_{r,m})_x \ne 0$ for some indices $r, m = (m_1, m_2, m_3)$, then there exist indices $k_1, \ldots, k_2 \in \{0, 1\}$ such that $k_1 + \cdots + k_n = m_1 + m_2 + m_3 \le n - r$, and $(\omega_{r,m})_x \wedge (\mathrm{d}_x x_1)^{k_1} \wedge \cdots \wedge (\mathrm{d}_x x_n)^{k_n} = \lambda\, \mathrm{d}_x x_1 \wedge \cdots \wedge \mathrm{d}_x x_n$, with $\lambda \ne 0$. In this case, in the right hand side of the formula (5.12) a term exists of the form

$$\begin{aligned}&\lambda'(\omega_{r,m})_x \wedge (\mathrm{d}x^1 \wedge \mathrm{d}y_1^1)^{h_1} \wedge (\mathrm{d}x^1 \wedge \mathrm{d}y_1^2)^{i_1} \wedge (\mathrm{d}x^1 \wedge \mathrm{d}y_1^3)^{j_1} \wedge \cdots \\ &\quad \wedge (\mathrm{d}x^n \wedge \mathrm{d}y_n^1)^{h_n} \wedge (\mathrm{d}x^n \wedge \mathrm{d}y_n^2)^{i_n} \wedge (\mathrm{d}x^n \wedge \mathrm{d}y_n^3)^{j_n},\end{aligned}$$

with $\lambda' \ne 0, h_1 + i_1 + j_1 = k_1, \ldots, h_n + i_n + j_n = k_n, h_1 + \cdots + h_n = m_1, i_1 + \cdots + i_n = m_2$, $j_1 + \cdots + j_n = m_3$. This term cannot cancel with any other term in (5.12) as once the indices $h_1, i_1, j_1, \ldots, h_n, i_n, j_n$ have been fixed, there is no other term containing

$$(\mathrm{d}y_1^1)^{h_1} \wedge (\mathrm{d}y_1^2)^{i_1} \wedge (\mathrm{d}y_1^3)^{j_1} \wedge \cdots \wedge (\mathrm{d}y_n^1)^{h_n} \wedge (\mathrm{d}y_n^2)^{i_n} \wedge (\mathrm{d}y_n^3)^{j_n}$$

as a factor, thus leading us to a contradiction. □

Remark 5.1. *As θ is aut P-invariant (see Remark 4.1), taking into account Proposition 4.1, from the formula (5.10) and the very definition of the forms ζ^a (see (4.3)), we conclude that η_4 is also* aut$P$*-invariant.*

6. Proof of Theorems 3.2 and 3.3

6.1. Proof of Theorem 3.2

We first state the following:

Lemma 6.1. *Let $\Omega_r = p^*\omega_r + p^*\omega_{r-4} \wedge \eta_4 + \cdots + p^*\omega_{r-4k} \wedge \eta_4^k$ be a form of degree r in $\mathcal{I}_{\mathrm{gau}\,P}(\mathcal{C}P)$, with $\omega_{r-4s} \in \Omega^{r-4s}(M)$, $0 \le s \le k = [r/4]$. If $\Omega_r = 0$, then $\omega_{r-4s} = 0$ for every s such that $r - 2s \le n$.*

Proof. Remark that if $r - 2s > n$, using formula (5.10) and Lemma 5.1, we have

$$q^*(p^*\omega_{r-4s} \wedge \eta_4^s) = (\tfrac{1}{4})^s \pi_1^*\omega_{r-4s} \wedge ((\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2)^s = 0,$$

which implies $p^*\omega_{r-4s} \wedge \eta_4^s = 0$. Hence the term $\omega_{r-4s} \wedge \eta_4^s$ does not appear in Ω_r. Now, assuming $\Omega_r = 0$ and pulling it back via q, we obtain

$$\begin{aligned} 0 = q^*\Omega_r &= \pi_1^*\omega_r + \tfrac{1}{4}\pi_1^*\omega_{r-4} \wedge ((\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2) + \cdots \\ &\quad + (\tfrac{1}{4})^k \pi_1^*\omega_{r-4k} \wedge ((\zeta^1)^2 + (\zeta^2)^2 + (\zeta^3)^2)^k. \end{aligned}$$

Again, by applying Lemma 5.1, we deduce $\pi_1^*\omega_{r-4s} = 0$, thus concluding the lemma. □

Let Ω_r be an aut$P$-invariant r-form on $\mathcal{C}(P)$. In particular, Ω_r is gau$P$-invariant and by virtue of Theorem 3.1, Ω_r can be written as

$$\Omega_r = p^*\omega_r + p^*\omega_{r-4} \wedge \eta_4 + \cdots + p^*\omega_{r-4k} \wedge \eta_4^k, \quad \omega_s \in \Omega^s(M).$$

Consider a trivialization $P|_U \cong U \times SU(2)$ on a coordinate domain $(U; x^1, \ldots, x^n)$ and let $X \in \mathrm{aut}\,\pi^{-1}(U)$ be the vector field given by formula (2.15) with $g^a = 0$ and arbitrary $f_j \in C^\infty(U)$. Then, as η_4 is an aut$P$-invariant form (see Remark 5.1), we have

$$0 = L_{X_C}\Omega_r = p^*L_{X'}\omega_r + p^*L_{X'}\omega_{r-4} \wedge \eta_4 + \cdots + p^*L_{X'}\omega_{r-4k} \wedge \eta_4^k,$$

where $X' = f_j(\partial/\partial x^j)$ is the p-projection of X onto U. Taking into account Lemma 6.1, this implies that $\forall X \in \mathfrak{X}(U)$, $L_X\omega_{r-4s} = 0$ if $r - 4s + 2s \le n$, and a form verifies this condition if and only if either it is a constant function in the case of 0-forms, or it identically vanishes in higher order degrees. Hence $\Omega_r = 0$ for $r \ne 4k$, and $\Omega_r = a\eta_4^k$, $a \in \mathbb{R}$, for $r = 4k$, thus proving the theorem.

6.2. Proof of Theorem 3.3

First we remark that η_4 is a closed form as follows from the formula (3.4) by a direct computation or else differentiating in (5.10) and taking into account that from the formula (4.3) we obtain

$$\sum_{a=1}^{3} \zeta^a \wedge \mathrm{d}\zeta^a = \mathfrak{S}_{123}(\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \theta^3 - \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \theta^3) = 0.$$

Moreover, as $p^* : H^4(M; \mathbb{R}) \to H^4(\mathcal{C}(P); \mathbb{R})$ is an isomorphism, for every connection Γ on P and every closed 4-form Ω_4 on $\mathcal{C}(P)$ we have $p^*[\sigma_\Gamma^* \Omega_4] = [\Omega_4]$. In particular $p^*[\sigma_\Gamma^* \eta_4] = [\eta_4]$. Then, pulling the formula (3.4) back via σ_Γ, according to (2.11) we obtain $\sigma_\Gamma^* \eta_4 = \det(\mathrm{d}A(\Gamma) + A(\Gamma) \wedge A(\Gamma))$, and pulling this equation back to the principal bundle P via π we have

$$\pi^*(\sigma_\Gamma^* \eta_4) = \det(\mathrm{d}\pi^* A(\Gamma) + \pi^* A(\Gamma) \wedge \pi^* A(\Gamma)) = \det(\Omega_\Gamma),$$

where Ω_Γ is the curvature form of Γ. We can thus finish by simply applying the definition of the Chern classes given in [13, XII.3].

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